

Localization effect for a spectral problem in a perforated domain with Fourier boundary conditions

V. Chiadò Piat ^{*} I. Pankratova [†] A. Piatnitski [‡]

Abstract

We consider a homogenization of elliptic spectral problem stated in a perforated domain, Fourier boundary conditions being imposed on the boundary of perforation. The presence of a locally periodic coefficient in the boundary operator gives rise to the effect of a localization of the eigenfunctions. Moreover, the limit behaviour of the lower part of the spectrum can be described in terms of an auxiliary harmonic oscillator operator. We describe the asymptotics of the eigenpairs and derive the estimates for the rate of convergence.

Keywords: Homogenization, spectral problem, localization.

1 Introduction

The paper deals with a spectral problem for a second order divergence form elliptic operator in a periodically perforated bounded domain in \mathbb{R}^d . Assuming that on the perforation border a homogeneous Fourier boundary condition is stated, and that the coefficient of the boundary operator is a function of "slow" argument, we arrive at the following eigenvalue problem

$$\begin{cases} -\operatorname{div}(a(x/\varepsilon)\nabla u^\varepsilon(x)) = \lambda^\varepsilon u^\varepsilon(x), & x \in \Omega_\varepsilon, \\ a(x/\varepsilon)\nabla u^\varepsilon(x) \cdot n = -q(x)u^\varepsilon(x), & x \in \Sigma_\varepsilon, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega; \end{cases} \quad (1.1)$$

here ε is a small positive parameter defined as a microstructure period.

We impose some natural regularity and connectedness conditions on the perforated domain Ω_ε , as well as usual periodicity and uniform ellipticity conditions on the matrix $a(y)$. These conditions are specified in detail in the next section.

Our crucial assumptions are

^{*}Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy (vchiado@polito.it)

[†]Narvik University College, Postbox 385, 8505 Narvik, Norway (iripan@hin.no)

[‡]Narvik University College, Postbox 385, 8505 Narvik, Norway; Lebedev Physical Institute RAS, Leninski ave., 53, 119991 Moscow, Russia (andrey@sci.lebedev.ru)

- $q \in C^2(\overline{\Omega})$, and $q(x) \geq q_0 > 0$ in $\overline{\Omega}$.
- The function q has only one global minimum point in $\overline{\Omega}$. The global minimum is attained at an interior point of Ω .
- The Hessian matrix $\partial^2 q / \partial x^2$ evaluated at the minimum point is positive definite.

Under the first two assumptions the localization phenomenon holds. Namely, for any $k \in \mathbb{N}$ the k -th eigenfunction of problem (1.1) is asymptotically localized, as $\varepsilon \rightarrow 0$, in a small neighbourhood of the minimum point. In particular, the properly normalized principal eigenfunction converges to a δ -function supported at the minimum point.

In the paper, assuming that all the above conditions are fulfilled, we construct the first two leading terms of the asymptotic expansions for the k -th eigenpair, $k = 1, 2, \dots$.

These asymptotic expansions have a number of interesting features. First of all, the mentioned expansions are in integer powers of $\varepsilon^{1/4}$. Then, the localization takes place in the scale $\varepsilon^{1/4}$. In this scale the leading term of the asymptotic expansion for the k -th eigenfunction proved to be the k -th eigenfunction of an auxiliary harmonic oscillator operator.

If $q \in C^3(\overline{\Omega})$, then we also obtain the estimates for the rate of convergence.

We suppose that q does not oscillate just for presentation simplicity. The techniques developed in the paper also apply to the case of locally periodic coefficients $q = q(x, x/\varepsilon)$, $a = a(x, x/\varepsilon)$ with $q(x, y)$ and $a(x, y)$ being periodic in y , see Remark 2.1 and Theorem 2.2 below.

Previously, the localization phenomenon in spectral problems has been observed in several mathematical works. In [1] the operator with a large locally periodic potential has been considered. The localization appeared due to the presence of a large factor in the potential and the fact that the operator coefficients depend on slow variable.

In [2] the Dirichlet spectral problem for the Laplacian in a thin 2D strip of slowly varying thickness has been studied. Here the localization has been observed in the vicinity of the point of maximum thickness. The large parameter is the first eigenvalue of 1D Laplacian in the cross-section. This eigenvalue grows to infinity because the thickness of the strip asymptotically vanishes.

In the mentioned works, under natural non-degeneracy conditions, the asymptotics of the eigenpairs was described in terms of the spectrum of an appropriate harmonic oscillator operator. However, the localization scale was of order $\sqrt{\varepsilon}$ with ε being the microscopic length scale.

The localization in the scale $\varepsilon^{1/4}$ that is observed in the present paper, is not standard. It should also be noted that although the operators in (1.1) do not contain a large parameter, such a parameter is presented implicitly because $(d - 1)$ -dimensional volume of the perforation surface tends to infinity.

The homogenization of spectral problem (1.1) with a constant or periodic functions a and q has been addressed in [7].

Spectral problems in perforated domains with Dirichlet and Neumann boundary condition at the perforation border are now well studied. There is a vast literature on the topic, see, for instance, [8], [6].

In the paper we combine asymptotic expansion techniques with various variational and compactness arguments and scaled trace and Poincaré type inequalities.

2 Problem statement

We start by describing the geometry of the domain. Let $K = [0, 1]^d$ and $E \subset \mathbb{R}^d$ be a K -periodic, open, connected set with a Lipschitz boundary Σ ; the complement $\mathbb{R}^d \setminus E$ is denoted by B . We also assume that $K \cap E$ is a connected set, and $K \cap B \Subset K$, so that $B = \mathbb{R}^d \setminus E$ consists of disjoint components. In what follows, $Y = K \cap E$ denotes the periodicity cell, and $\Sigma^0 = K \cap \partial B = K \cap \Sigma$ the boundary of the inclusion. The symbols $|Y|_d$ and $|\Sigma^0|_{d-1}$ stand for the measures of Y and the $(d-1)$ -dimensional surface measure of Σ^0 , respectively.

For every $i \in \mathbb{Z}^d$ we denote $Y_\varepsilon^i = \varepsilon(i + Y)$, $\Sigma_\varepsilon^i = \varepsilon \Sigma \cap Y_\varepsilon^i$, and $B_\varepsilon^i = \varepsilon B \cap Y_\varepsilon^i$. Given Ω , a bounded domain in \mathbb{R}^d with a Lipschitz boundary $\partial\Omega$, we introduce the perforated domain

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{i \in I_\varepsilon} B_\varepsilon^i, \quad I_\varepsilon = \{i \in \mathbb{Z}^d : Y_\varepsilon^i \subset \Omega\}.$$

Notice that Ω_ε remains connected, the perforation does not intersect the boundary $\partial\Omega$, and

$$\partial\Omega_\varepsilon = \partial\Omega \bigcup \Sigma_\varepsilon, \quad \Sigma_\varepsilon = \bigcup_{i \in I_\varepsilon} \Sigma_\varepsilon^i.$$

In the perforated domain Ω_ε we consider the following spectral problem:

$$\begin{cases} -\operatorname{div}(a^\varepsilon(x) \nabla u^\varepsilon(x)) = \lambda^\varepsilon u^\varepsilon(x), & x \in \Omega_\varepsilon, \\ a^\varepsilon(x) \nabla u^\varepsilon(x) \cdot n = -q(x) u^\varepsilon(x), & x \in \Sigma_\varepsilon, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega. \end{cases} \quad (2.1)$$

Here ε is a small positive parameter, $a^\varepsilon(x) = a(x/\varepsilon)$ with $a(y)$ being a $d \times d$ matrix, n is an outward unit normal; the usual scalar product in \mathbb{R}^d is denoted by " \cdot ".

In the sequel we assume that the following conditions hold true:

(H1) $a(y)$ is a real symmetric $d \times d$ matrix satisfying the uniform ellipticity condition

$$\sum_{i,j=1}^d a_{ij}(y) \xi_i \xi_j \geq \Lambda |\xi|^2, \quad \xi \in \mathbb{R}^d,$$

for some $\Lambda > 0$.

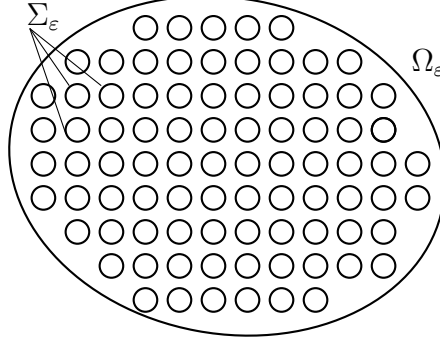


Figure 1: Domain Ω_ε

- (H2) The coefficients $a_{ij}(y)$ are Y -periodic and, moreover, $a_{ij}(y) \in L^\infty(\mathbb{R}^d)$.
- (H3) The function $q(x) \in C^3(\mathbb{R}^d)$ is positive.
- (H4) The function $q(x)$ has a unique global minimum attained at $x = 0 \in \Omega$.
Moreover, in the vicinity of $x = 0$

$$q(x) = q(0) + \frac{1}{2} x^T H(q) x + o(|x|^2),$$

with the positive definite Hessian matrix $H(q)$.

It is convenient to introduce the notation

$$H_0^1(\Omega_\varepsilon, \partial\Omega) = \{u \in H^1(\Omega_\varepsilon) : u = 0 \text{ on } \partial\Omega\}.$$

The weak formulation of spectral problem (2.1) reads: find $\lambda^\varepsilon \in \mathbb{C}$ (eigenvalues) and $u^\varepsilon \in H_0^1(\Omega_\varepsilon, \partial\Omega)$, $u^\varepsilon \neq 0$, such that

$$\int_{\Omega_\varepsilon} a^\varepsilon \nabla u^\varepsilon \cdot \nabla v \, dx + \int_{\Sigma_\varepsilon} q u^\varepsilon v \, d\sigma = \lambda^\varepsilon \int_{\Omega_\varepsilon} u^\varepsilon v \, dx, \quad v \in H_0^1(\Omega). \quad (2.2)$$

Lemma 2.1. *For any $\varepsilon > 0$, the spectrum of problem (2.2) is real and consists of a countable set of points*

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon \leq \dots \leq \lambda_j^\varepsilon \leq \dots \rightarrow +\infty.$$

Every eigenvalue has a finite multiplicity. The corresponding eigenfunctions normalized by

$$\int_{\Omega_\varepsilon} u_i^\varepsilon u_j^\varepsilon \, dx = \delta_{ij},$$

form an orthonormal basis in $L^2(\Omega_\varepsilon)$.

We omit the proof of Lemma 2.1 which is classical.

Under the assumptions (H1)-(H4) we study the asymptotic behaviour of eigenpairs $(\lambda^\varepsilon, u^\varepsilon)$, as $\varepsilon \rightarrow 0$.

To avoid excessive technicalities for the moment, we state our main result in a slightly reduced form, without specifying the rate of convergence. For the detailed formulation of the main result see Theorem 3.2.

Theorem 2.1. *Let conditions (H1)-(H4) be fulfilled. If $(\lambda_j^\varepsilon, u_j^\varepsilon)$ stands for the j th eigenpair of problem (2.1), then for any j , the following representation takes place:*

$$\lambda_j^\varepsilon = \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + \frac{\mu_j^\varepsilon}{\sqrt{\varepsilon}}, \quad u_j^\varepsilon(x) = v_j^\varepsilon\left(\frac{x}{\varepsilon^{1/4}}\right),$$

where $(\mu_j^\varepsilon, v_j^\varepsilon(z))$ are such that

- μ_j^ε converges, as $\varepsilon \rightarrow 0$, to the j th eigenvalue μ_j of the effective spectral problem

$$-\operatorname{div}(a^{\text{eff}} \nabla v) + (z^T Q z) v = \mu v, \quad v \in L^2(\mathbb{R}^d), \quad (2.3)$$

where a^{eff} is a positive definite matrix (see (3.19)); Q is defined by

$$Q = \frac{1}{2} \frac{|\Sigma^0|_{d-1}}{|Y|_d} H(q),$$

with $H(q)$ being the Hessian matrix of q at $x = 0$.

- If μ_j is a simple eigenvalue, then, for small ε , μ_j^ε is also simple, and the convergence of the corresponding eigenfunctions (extended to the whole \mathbb{R}^d) holds

$$\|v_j^\varepsilon - v_j\|_{L^2(\mathbb{R}^d)} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

Remark 2.1. *Theorem 2.1 can be generalized to the case of locally periodic coefficients in (2.1).*

Namely, let us consider the following problem:

$$\begin{cases} -\operatorname{div}(a^\varepsilon(x) \nabla u^\varepsilon(x)) = \lambda^\varepsilon u^\varepsilon(x), & x \in \Omega_\varepsilon, \\ a^\varepsilon(x) \nabla u^\varepsilon(x) \cdot n = -q^\varepsilon(x) u^\varepsilon(x), & x \in \Sigma_\varepsilon, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega \end{cases} \quad (2.4)$$

with

$$a^\varepsilon(x) = a(x, x/\varepsilon), \quad q^\varepsilon(x) = q(x, x/\varepsilon).$$

Assume that

- $a_{ij}(x, y)$ and $q(x, y)$ are Y -periodic in y functions such that $a_{ij}(x, y), q(x, y) \in C^{2,\alpha}(\mathbb{R}^d; C^\alpha(\overline{Y}))$ with some $\alpha > 0$.
- The matrix $a(x, y)$ satisfies the uniform ellipticity condition.
- The local average of q defined by

$$\bar{q}(x) = \frac{1}{|\Sigma^0|_{d-1}} \int_{\Sigma^0} q(x, y) d\sigma_y,$$

admits its global minimum at $x = 0$.

- In the vicinity of $x = 0$

$$\bar{q}(x) = \bar{q}(0) + \frac{1}{2} x^T H(\bar{q}) x + o(|x|^2)$$

with the positive definite Hessian matrix $H(\bar{q})$.

- $x = 0$ is the only global minimum point of \bar{q} in $\bar{\Omega}$.

Then the following convergence result holds.

Theorem 2.2. *If $(\lambda_j^\varepsilon, u_j^\varepsilon)$ stands for the j th eigenpair of problem (2.4), then for any j , the following representation takes place:*

$$\lambda_j^\varepsilon = \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \bar{q}(0) + \frac{\mu_j^\varepsilon}{\sqrt{\varepsilon}}, \quad u_j^\varepsilon(x) = v_j^\varepsilon\left(\frac{x}{\varepsilon^{1/4}}\right),$$

where $(\mu_j^\varepsilon, v_j^\varepsilon(z))$ are such that

- μ_j^ε converges, as $\varepsilon \rightarrow 0$, to the j th eigenvalue μ_j of the effective spectral problem

$$-\operatorname{div}(a^{\text{eff}} \nabla v) + (z^T P z) v = \mu v, \quad v \in L^2(\mathbb{R}^d), \quad (2.5)$$

where

$$P = \frac{1}{2} \frac{|\Sigma^0|_{d-1}}{|Y|_d} H(\bar{q})$$

and a^{eff} is a positive definite matrix defined by

$$a_{ij}^{\text{eff}} = \frac{1}{|Y|_d} \int_Y a_{ik}(0, \zeta) (\delta_{kj} + \partial_k N_j(\zeta)) d\zeta,$$

with the functions N_j solving auxiliary cell problems

$$\begin{cases} -\operatorname{div}_\zeta(a(0, \zeta) \nabla_\zeta N_k(\zeta)) = \operatorname{div}_\zeta a_{\cdot k}(0, \zeta), & k = 1, \dots, d, \quad \zeta \in Y, \\ a(0, \zeta) \nabla_\zeta N_k \cdot n = -a_{ik}(0, \zeta) n_i, & \zeta \in \Sigma^0, \\ N_k(\zeta) \in H_{\#}^1(Y), \end{cases}$$

- If μ_j is a simple eigenvalue, then, for small ε , μ_j^ε is also simple, and the convergence of the corresponding eigenfunctions (extended to the whole \mathbb{R}^d) holds

$$\|v_j^\varepsilon - v_j\|_{L^2(\mathbb{R}^d)} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

3 Proof of Theorem 2.1

3.1 Preliminaries. Estimates for λ_1^ε

In this section we estimate the first eigenvalue λ_1^ε of problem (2.1). To this end we use the variational representation for λ_1^ε . Let us recall that, due to the classical min-max principle (see, for example, [4]),

$$\lambda_1^\varepsilon = \inf_{v \in H_0^1(\Omega_\varepsilon, \partial\Omega)} \frac{\int_{\Omega_\varepsilon} a^\varepsilon \nabla v \cdot \nabla v dx + \int_{\Sigma_\varepsilon} q(v)^2 d\sigma}{\|v\|_{L^2(\Omega_\varepsilon)}^2}. \quad (3.1)$$

Lemma 3.1. *The first eigenvalue of the spectral problem (2.1) satisfies the estimate*

$$\frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + O(1) \leq \lambda_1^\varepsilon \leq \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + O(\varepsilon^{-1/2}), \quad \varepsilon \rightarrow 0.$$

Proof. We start by proving the estimate from below. By (3.1),

$$\lambda_1^\varepsilon \geq \inf_{\substack{v \in H_0^1(\Omega_\varepsilon, \partial\Omega) \\ \|v\|_{L^2(\Omega_\varepsilon)}=1}} \left\{ \int_{\Omega_\varepsilon} a^\varepsilon \nabla v \cdot \nabla v \, dx + q(0) \int_{\Sigma_\varepsilon} (v)^2 \, d\sigma \right\}.$$

The last infimum is attained on the first eigenfunction of the following spectral problem

$$\begin{cases} -\operatorname{div}(a^\varepsilon(x) \nabla w^\varepsilon(x)) = \nu^\varepsilon w^\varepsilon(x), & x \in \Omega_\varepsilon, \\ a^\varepsilon(x) \nabla w^\varepsilon(x) \cdot n = -q(0) w^\varepsilon(x), & x \in \Sigma_\varepsilon, \\ w^\varepsilon(x) = 0, & x \in \partial\Omega. \end{cases}$$

It has been proven in [7] that the first eigenvalue of this problem admits the following asymptotics:

$$\nu_1^\varepsilon = \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + O(1), \quad \varepsilon \rightarrow 0.$$

Thus,

$$\lambda_1^\varepsilon \geq \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + O(1), \quad \varepsilon \rightarrow 0.$$

We proceed to the derivation of the upper bound for λ_1^ε . Choosing $v \in C_0^\infty(\Omega)$ as a test function in (3.1), one can obtain a rough estimate

$$\lambda_1^\varepsilon \leq \tilde{C} \varepsilon^{-1}, \quad (3.2)$$

with a constant \tilde{C} independent of ε . To specify \tilde{C} one should choose a "smarter" test function. Let us take $v \in C_0^\infty(\mathbb{R}^d)$, $\|v\|_{L^2(\mathbb{R}^d)} = 1$, and choose $v(x/\varepsilon^\alpha)$ as a test function in (3.1), $0 < \alpha < 1/2$. Note that if $\operatorname{supp} v \subset B_R(0)$, for some $R > 0$, then $\operatorname{supp} v(x/\varepsilon^\alpha) \subset B_{\varepsilon^\alpha R}(0)$. Then we obtain

$$\lambda_1^\varepsilon \leq \frac{\int_{\Sigma_\varepsilon} q(x) \left| v\left(\frac{x}{\varepsilon^\alpha}\right) \right|^2 d\sigma + O(\varepsilon^{-2\alpha} \varepsilon^{d\alpha})}{\int_{\Omega_\varepsilon} \left| v\left(\frac{x}{\varepsilon^\alpha}\right) \right|^2 dx}.$$

Taking into account assumption **(H4)** and using Lemma 4.1, one has

$$\lambda_1^\varepsilon \leq \frac{\frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) \int_{\Omega_\varepsilon} \left| v\left(\frac{x}{\varepsilon^\alpha}\right) \right|^2 dx + O(\varepsilon^{2\alpha-1} \varepsilon^{d\alpha}) + O(\varepsilon^{-2\alpha} \varepsilon^{d\alpha})}{\int_{\Omega_\varepsilon} \left| v\left(\frac{x}{\varepsilon^\alpha}\right) \right|^2 dx}.$$

Notice that the best estimate is obtained for $\alpha = 1/4$. Finally,

$$\lambda_1^\varepsilon \leq \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + O(\varepsilon^{-1/2}), \quad \varepsilon \rightarrow 0. \quad (3.3)$$

□

Remark 3.1. When deriving the upper bound for λ_1^ε , we used a test function which is concentrated at $x = 0$. Namely, the test function of the form $v(\varepsilon^{-1/4}x)$. This observation turns out very helpful for the construction of the asymptotics of eigenpairs $(\lambda^\varepsilon, u^\varepsilon)$.

The next definition explains the notion of concentration.

Definition 3.1. We say that a family $\{w_\varepsilon(x)\}_{\varepsilon>0}$ with $0 < c_1 \leq \|w_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c_2$ is concentrated at x_0 , as $\varepsilon \rightarrow 0$, if for any $\gamma > 0$ there is $\varepsilon_0 > 0$ such that

$$\int_{\Omega_\varepsilon \setminus B_\gamma(x_0)} |w_\varepsilon|^2 dx < \gamma, \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

Here $B_\gamma(x_0)$ is a ball of radius γ centered at x_0 .

Lemma 3.2. The first eigenfunction u_1^ε of problem (2.1) is concentrated in the sense of Definition 3.1 at the minimum point of $q(x)$, that is at $x = 0$.

Proof. Assume that u_1^ε , normalized by $\|u_1^\varepsilon\|_{L^2(\Omega_\varepsilon)} = 1$, is not concentrated at $x = 0$. Then, there exists $\gamma > 0$ such that, for any ε_0 , we have

$$\int_{\Omega_\varepsilon \setminus B_\gamma(0)} |u_1^\varepsilon|^2 dx > \gamma \quad (3.4)$$

for some $\varepsilon < \varepsilon_0$.

Estimate (3.2) together with (3.1) imply the estimate

$$\int_{\Omega_\varepsilon} |\nabla u_1^\varepsilon|^2 dx \leq C \varepsilon^{-1}.$$

Then, using Lemma 4.1, we obtain

$$\begin{aligned} \lambda_1^\varepsilon &= \int_{\Omega_\varepsilon} a^\varepsilon \nabla u_1^\varepsilon \cdot \nabla u_1^\varepsilon dx + \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \int_{\Omega_\varepsilon} q |u_1^\varepsilon|^2 dx + O(\varepsilon^{-1/2}) \\ &\geq \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \min_{\Omega_\varepsilon \setminus B_\gamma(0)} q \int_{\Omega_\varepsilon \setminus B_\gamma(0)} |u_1^\varepsilon|^2 dx \\ &\quad + \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \left\{ q(0) \int_{\Omega_\varepsilon \cap B_\gamma(0)} |u_1^\varepsilon|^2 dx + \int_{\Omega_\varepsilon \cap B_\gamma(0)} (q(x) - q(0)) |u_1^\varepsilon|^2 dx \right\} + O(\varepsilon^{-1/2}) \end{aligned}$$

Since $x = 0$ is the global minimum point of $q(x)$, then

$$\lambda_1^\varepsilon \geq \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \left\{ \min_{\Omega_\varepsilon \setminus B_\gamma(0)} q \int_{\Omega_\varepsilon \setminus B_\gamma(0)} |u_1^\varepsilon|^2 dx + q(0) \int_{\Omega_\varepsilon \cap B_\gamma(0)} |u_1^\varepsilon|^2 dx \right\} + O(\varepsilon^{-1/2}).$$

By (3.4),

$$\lambda_1^\varepsilon \geq \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \left(\min_{\Omega_\varepsilon \setminus B_\gamma(0)} q - q(0) \right) \gamma + O(\varepsilon^{-1/2}), \quad (3.5)$$

that contradicts (3.3). Lemma is proved. \square

Remark 3.2. *The min-max principle allows us to compare the eigenvalues of Dirichlet, Neumann and Fourier spectral problems. Namely, denote by $\lambda_{D,k}^\varepsilon$ the k th eigenvalue of the Dirichlet problem ($u^\varepsilon = 0$ on Σ_ε), and by $\lambda_{N,k}^\varepsilon$ the k th eigenvalue of the Neumann problem (the case $q = 0$ in (2.1)). Then, one can see that*

$$\lambda_{N,k}^\varepsilon \leq \lambda_k^\varepsilon \leq \lambda_{D,k}^\varepsilon, \quad k = 1, 2, \dots \quad (3.6)$$

It is well-known (see [8]) that $\lambda_{N,k}^\varepsilon = O(1)$ and $\lambda_{D,k}^\varepsilon = O(\varepsilon^{-2})$, $\varepsilon \rightarrow 0$. Lemma 3.1 specifies estimate (3.6) for the first eigenvalue λ_1^ε .

3.2 Change of unknowns. Rescaled problem

For brevity, we denote

$$\varkappa(x) = \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(x), \quad Q = \frac{1}{2} \frac{|\Sigma^0|_{d-1}}{|Y|_d} H(q),$$

where $H(q)$ is the Hessian matrix of q at $x = 0$.

Note that Lemma 3.1 suggests to study the asymptotics of $(\lambda_k^\varepsilon - \varepsilon^{-1} \varkappa(0))$, rather than of λ_k^ε itself. On the other hand, when deriving the upper bound in Lemma 3.1, we used the test function $v(x/\varepsilon^{1/4})$, which allowed us to get the "optimal" estimate. Bearing in mind these two ideas, we first subtract $\varepsilon^{-1} \varkappa(0) u^\varepsilon(x)$ from both sides of the equation in (2.1), and then make the change of variables $z = \varepsilon^{-1/4} x$ in (2.1). Then, the rescaled problem is stated in the domain

$$\widetilde{\Omega}_\varepsilon = \varepsilon^{-1/4} \Omega_\varepsilon, \quad \widetilde{\Sigma}_\varepsilon = \varepsilon^{-1/4} \Sigma_\varepsilon,$$

and takes the form

$$\begin{cases} -\operatorname{div}(a^\varepsilon(z) \nabla v^\varepsilon(z)) - \frac{\varkappa(0)}{\sqrt{\varepsilon}} v^\varepsilon = \mu^\varepsilon v^\varepsilon(x), & z \in \widetilde{\Omega}_\varepsilon, \\ a^\varepsilon(z) \nabla v^\varepsilon(z) \cdot n = -\varepsilon^{1/4} q(\varepsilon^{1/4} z) v^\varepsilon(z), & z \in \widetilde{\Sigma}_\varepsilon, \\ v^\varepsilon(z) = 0, & z \in \varepsilon^{-1/4} \partial\Omega. \end{cases} \quad (3.7)$$

Here

$$v^\varepsilon(z) = u^\varepsilon(\varepsilon^{1/4} z), \quad a^\varepsilon(z) = a\left(\frac{z}{\varepsilon^{3/4}}\right), \quad \mu^\varepsilon = \sqrt{\varepsilon} \left(\lambda^\varepsilon - \frac{\varkappa(0)}{\varepsilon} \right). \quad (3.8)$$

The weak formulation of problem (3.1) reads: find $(\mu^\varepsilon, v^\varepsilon) \in \mathbb{R} \times H_0^1(\widetilde{\Omega}_\varepsilon, \varepsilon^{-1/4} \partial\Omega)$, $v^\varepsilon \neq 0$, such that

$$W^\varepsilon(v^\varepsilon, w) = \mu^\varepsilon (v^\varepsilon, w)_{L^2(\widetilde{\Omega}_\varepsilon)}, \quad \forall w \in H_0^1(\widetilde{\Omega}_\varepsilon, \varepsilon^{-1/4} \partial\Omega). \quad (3.9)$$

Here the bilinear form $W^\varepsilon(u, v)$ is given by

$$W^\varepsilon(u, v) = \int_{\widetilde{\Omega}_\varepsilon} a^\varepsilon \nabla u \cdot \nabla v \, dz - \frac{\varkappa(0)}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega}_\varepsilon} u v \, dz + \varepsilon^{1/4} \int_{\widetilde{\Sigma}_\varepsilon} q(\varepsilon^{1/4} z) u v \, d\sigma_z. \quad (3.10)$$

Remark 3.3. About the extension operator For all sufficiently small ε , there exists an extension operator

$$P^\varepsilon : H_0^1(\widetilde{\Omega}_\varepsilon, \varepsilon^{-1/4}\partial\Omega) \rightarrow H_0^1(\varepsilon^{-1/4}\Omega)$$

such that

$$\|P^\varepsilon v\|_{L^2(\varepsilon^{-1/4}\Omega)} \leq C \|v\|_{L^2(\widetilde{\Omega}_\varepsilon)}, \quad \|\nabla(P^\varepsilon v)\|_{L^2(\varepsilon^{-1/4}\Omega)} \leq C \|\nabla v\|_{L^2(\widetilde{\Omega}_\varepsilon)},$$

where C is a constant independent of ε .

Moreover, the obtained extended function (for which we keep the same notation) can be extended by zero to the whole \mathbb{R}^d , outward the boundary $\varepsilon^{-1/4}\partial\Omega$.

Proposition 3.1. The spectrum of problem (3.9) is real, discrete and consists of a countable set of points

$$0 < \mu_1^\varepsilon < \mu_2^\varepsilon \leq \dots \leq \mu_j^\varepsilon \leq \dots \rightarrow +\infty.$$

The corresponding eigenfunctions can be normalized by

$$W^\varepsilon(v_i^\varepsilon, v_j^\varepsilon) = \delta_{ij}, \quad (3.11)$$

with $W^\varepsilon(u, v)$ defined by (3.10).

Proof. For any fixed $\varepsilon > 0$, the bilinear form $W^\varepsilon(\cdot, \cdot)$ defines an equivalent scalar product in $H_0^1(\widetilde{\Omega}_\varepsilon, \varepsilon^{-1/4}\partial\Omega)$. For brevity, we denote

$$H_{0,W}^1(\widetilde{\Omega}_\varepsilon) = \{w \in H_0^1(\widetilde{\Omega}_\varepsilon, \varepsilon^{-1/4}\partial\Omega) : \|w\|_{\varepsilon,W}^2 = W^\varepsilon(w, w) < \infty\}. \quad (3.12)$$

Let $G^\varepsilon : L^2(\widetilde{\Omega}_\varepsilon) \rightarrow H_{0,W}^1(\widetilde{\Omega}_\varepsilon)$ be the operator defined as follows:

$$W^\varepsilon(G^\varepsilon f, w) = (f, w)_{L^2(\widetilde{\Omega}_\varepsilon)}, \quad w \in H_{0,W}^1(\widetilde{\Omega}_\varepsilon, \varepsilon^{-1/4}\partial\Omega). \quad (3.13)$$

Obviously, G^ε is a positive, bounded (uniformly in ε), self-adjoint operator. Since $H_{0,W}^1(\widetilde{\Omega}_\varepsilon)$, for each fixed ε , is compactly embedded into $L^2(\widetilde{\Omega})$, then G^ε is compact as an operator from $L^2(\widetilde{\Omega})$ ($H_{0,W}^1(\widetilde{\Omega}_\varepsilon)$) into itself.

Thus, the spectrum $\sigma(G^\varepsilon)$ is a countable set of points in \mathbb{R} which does not have any accumulation points except for zero. Every nonzero eigenvalue has finite multiplicity. To complete the proof of the proposition, it is left to notice that in terms of the operator G^ε the eigenvalue problem (3.7) takes the form

$$G^\varepsilon v^\varepsilon = \frac{1}{\mu^\varepsilon} v^\varepsilon.$$

□

We proceed with auxiliary technical results that will be useful in the sequel. Define the following norms in $H^1(\widetilde{\Omega}_\varepsilon)$:

$$\|v\|_{\varepsilon,W}^2 = \int_{\widetilde{\Omega}_\varepsilon} a^\varepsilon \nabla v \cdot \nabla v \, dz - \frac{\varkappa(0)}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega}_\varepsilon} |v|^2 \, dz + \varepsilon^{1/4} \int_{\widetilde{\Sigma}_\varepsilon} q(\varepsilon^{1/4}z) |v|^2 \, d\sigma_z;$$

$$\|v\|_{\varepsilon, \kappa}^2 = \int_{\widetilde{\Omega}_\varepsilon} a^\varepsilon \nabla v \cdot \nabla v \, dz + \frac{1}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega}_\varepsilon} (\kappa(\varepsilon^{1/4} z) - \kappa(0)) |v|^2 \, dz; \quad (3.14)$$

$$\|v\|_{\varepsilon, Q}^2 = \int_{\widetilde{\Omega}_\varepsilon} a^\varepsilon \nabla v \cdot \nabla v \, dz + \int_{\widetilde{\Omega}_\varepsilon} (z^T Q z) |v|^2 \, dz.$$

Lemma 3.3. *The norms $\|\cdot\|_{\varepsilon, W}$, $\|\cdot\|_{\varepsilon, \kappa}$ and $\|\cdot\|_{\varepsilon, Q}$ are equivalent. Moreover,*

$$\begin{aligned} C_1 \|v\|_{\varepsilon, \kappa}^2 &\leq \|v\|_{\varepsilon, W}^2 \leq C_2 \|v\|_{\varepsilon, \kappa}^2; \\ C_3 \|v\|_{\varepsilon, \kappa}^2 &\leq \|v\|_{\varepsilon, Q}^2 \leq C_4 \|v\|_{\varepsilon, \kappa}^2, \end{aligned} \quad (3.15)$$

with constants C_1, C_2, C_3 and C_4 that do not depend on ε .

Proof. Indeed, by Lemma 4.2 and by the Poincaré inequality,

$$|||v||_{\varepsilon, W}^2 - \|v\|_{\varepsilon, \kappa}^2| \leq C \varepsilon^{1/4} \|v\|_{L^2(\widetilde{\Omega}_\varepsilon)} \|\nabla v\|_{L^2(\widetilde{\Omega}_\varepsilon)} \leq C_1 \varepsilon^{1/4} \|v\|_{\varepsilon, \kappa}^2$$

and, thus, the first inequality in (3.15) holds for sufficiently small ε .

The second inequality follows easily from the hypothesis **(H4)** and Lemma 4.3. \square

Remark 3.4. *If $v \in H^1(\widetilde{\Omega}_\varepsilon)$ decays exponentially, namely,*

$$\|v\|_{L^2(\mathbb{R}^d \setminus B_R(0))} \leq M e^{-\gamma_0 R},$$

for some constant M , then the norms defined in Lemma 3.3 are asymptotically close. In particular, the following estimate holds:

$$|||v||_{\varepsilon, W}^2 - \|v\|_{\varepsilon, Q}^2| \leq C \varepsilon^{1/4}$$

with the constant $C = C(M, \gamma_0)$ independent of ε .

Lemma 3.4. *Let μ_1^ε be the first eigenvalue of the spectral problem (3.7). Then there exist two positive constants C_1 and C_2 such that*

$$C_1 \leq \mu_1^\varepsilon \leq C_2.$$

Proof. The upper bound follows from (3.8) and Lemma 3.1. The lower bound is the consequence of the boundedness of the operator G^ε (see the proof of Proposition 3.1). \square

3.2.1 Formal asymptotic expansion for the rescaled problem

Following the classical asymptotic expansion method and bearing in mind Lemma 3.4, we seek for a solution of problem (3.7) in the form of asymptotic series

$$\begin{aligned} \mu^\varepsilon &= \mu + \varepsilon^{1/4} \mu_{\frac{1}{4}} + \varepsilon^{1/2} \mu_{\frac{1}{2}} + \cdots, \\ v^\varepsilon &= v(z) + \varepsilon^{1/4} v_{\frac{1}{4}}(z, \zeta) + \varepsilon^{1/2} v_{\frac{1}{2}}(z, \zeta) + \varepsilon^{3/4} v_{\frac{3}{4}}(z, \zeta) + \cdots, \quad \zeta = \frac{z}{\varepsilon^{3/4}}, \end{aligned} \quad (3.16)$$

where the functions $v_{\frac{k}{4}}(z, \zeta)$ are Y -periodic in ζ , $k = 1, 2, \dots$.

Substituting ansätze (3.16) into (3.7) and collecting the terms of order $\varepsilon^{-5/4}$ and ε^{-1} in the equation, and of order $\varepsilon^{-1/2}$, $\varepsilon^{-1/4}$ in the boundary condition, we see that the functions $v_{\frac{1}{4}}$ and $v_{\frac{1}{2}}$ do not depend on ζ . Then, collecting the terms of order $\varepsilon^{-3/4}$, we obtain that

$$v_{\frac{3}{4}}(z, \zeta) = N_k(\zeta) \partial_k v(z) + w_3(z),$$

where the vector function $N(\zeta)$ solves the problem

$$\begin{cases} -\operatorname{div}_{\zeta}(a(\zeta) \nabla_{\zeta} N_k(\zeta)) = \operatorname{div}_{\zeta} a_{\cdot k}(\zeta), & k = 1, \dots, d, \quad \zeta \in Y, \\ a \nabla_{\zeta} N_k \cdot n = -a_{ik} n_i, & \zeta \in \Sigma^0, \\ N_k(\zeta) \in H_{\#}^1(Y), \end{cases} \quad (3.17)$$

The effective spectral problem comes out while collecting the terms of order ε^0 and writing the compatibility condition for the resulting problem. It reads

$$-\operatorname{div}(a^{\text{eff}} \nabla v) + (z^T Q z) v = \mu v, \quad v \in L^2(\mathbb{R}^d), \quad (3.18)$$

where a^{eff} is given by

$$a_{ij}^{\text{eff}} = \frac{1}{|Y|_d} \int_Y a_{ik}(y) (\delta_{kj} + \partial_k N_j) dy. \quad (3.19)$$

The effective problem describes the eigenvalues and eigenfunctions of d -dimensional harmonic oscillator. In \mathbb{R}^1 an explicit solution can be given in terms of Hermite polynomials. In the case under consideration we prove the following statement that characterizes the spectrum of problem (3.18).

Lemma 3.5. *The spectrum of the effective problem (3.18) is real and discrete*

$$0 < \mu_1 < \mu_2 \leq \dots \leq \mu_j \dots \rightarrow +\infty.$$

The corresponding eigenfunctions $v_j(z)$ can be normalized by

$$(v_i, v_j)_Q \equiv \int_{\mathbb{R}^d} a^{\text{eff}} \nabla v_i \cdot \nabla v_j dz + \int_{\mathbb{R}^d} (z^T Q z) v_i v_j dz = \delta_{ij}. \quad (3.20)$$

We omit the proof of Lemma 3.5 which is classical.

It is well known that the eigenfunctions of the harmonic oscillator operator have the form

$$v_j(z) = P_{j-1}(z) e^{-z^T R z}, \quad R = \frac{\sqrt{2} Q^{1/2} (a^{\text{eff}})^{-1/2}}{2}, \quad (3.21)$$

where $P_k(z)$ is a polynomial of degree k .

To summarize, the formal asymptotic expansion for v^ε takes the form

$$v(z) + \varepsilon^{3/4} N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v(z),$$

where v is an eigenfunction of the limit spectral problem (3.18), N is a periodic vector function solving (3.17).

Notice that we can neglect the summands $v_{\frac{1}{4}}$ and $v_{\frac{1}{2}}$ since they do not depend on the fast variable ζ , and thus, their H^1 -norm is of order $\varepsilon^{1/4}$.

3.2.2 Justification

Denote $J(j) = \min\{i \in \mathbb{Z}^+ : \mu_i = \mu_j\}$, and let κ_j be the multiplicity of the j th eigenvalue μ_j of the harmonic oscillator operator (3.18).

The main goal of this section is to prove the following theorem.

Theorem 3.1. *Let hypotheses (H1)-(H4) be fulfilled. If $(\mu_p^\varepsilon, v_p^\varepsilon)$ stands for p th eigenpair of problem 3.7, then the following statements hold true:*

1. *For each $j = 1, 2, \dots$, there exist $\varepsilon_j > 0$ and a constant c_j such that the eigenvalue μ_j^ε of problem (3.7) satisfies the inequality*

$$|\mu_j^\varepsilon - \mu_j| \leq c_j \varepsilon^{1/4}, \quad \varepsilon \in (0, \varepsilon_j),$$

where μ_j is an eigenvalue of the harmonic oscillator operator (3.18).

2. *There exists a unitary $\kappa_j \times \kappa_j$ matrix β^ε such that*

$$\left\| v_p^\varepsilon - \sum_{k=J(j)}^{J(j)+\kappa_j-1} \beta_{pk}^\varepsilon \tilde{V}_k^\varepsilon \right\|_{\varepsilon, Q} \leq C_j \varepsilon^{1/4}, \quad p = J(j), \dots, J(j) + \kappa_j - 1, \quad (3.22)$$

where

$$\tilde{V}_k^\varepsilon = v_k(z) + \varepsilon^{3/4} N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v_k(z). \quad (3.23)$$

Here the vector function $N(\zeta)$ solve problem (3.17); eigenfunctions $v_k(z)$ of the limit problem are defined in (3.18); the norm $\|\cdot\|_{\varepsilon, Q}$ is defined just before Lemma 3.3.

Moreover, almost eigenfunctions $\{\tilde{V}_k^\varepsilon\}$ satisfy the following orthogonality and normalization condition:

$$\left| \int_{\tilde{\Omega}_\varepsilon} a^\varepsilon \nabla \tilde{V}_k^\varepsilon \cdot \nabla \tilde{V}_m^\varepsilon dz + \int_{\tilde{\Omega}_\varepsilon} (z^T Q z) \tilde{V}_k^\varepsilon \tilde{V}_m^\varepsilon dz - \delta_{km} \right| \leq C \varepsilon^{1/4}. \quad (3.24)$$

Proof. The justification procedure will rely on Vishik's lemma about "almost eigenvalues and eigenfunctions" (see, for example, [9] and [5], p. 319, Lemma 1.5). For the reader's convenience, we formulate the mentioned result.

Lemma 3.6. *Given a self-adjoint operator $\mathcal{K}^\varepsilon : \mathcal{H} \rightarrow \mathcal{H}$ with a discrete spectrum, let $\nu \in \mathbb{R}$ and $v \in \mathcal{H}$ be such that*

$$\|v\|_{\mathcal{H}} = 1, \quad \delta \equiv \|\mathcal{K}^\varepsilon v - \nu v\|_{\mathcal{H}} < |\nu|.$$

Then there exists an eigenvalue μ_l^ε of the operator \mathcal{K}^ε such that

$$|\mu_l^\varepsilon - \nu| \leq \delta.$$

Moreover, for any $\delta_1 \in (\delta, |\nu|)$ there exist $\{a_j^\varepsilon\} \in \mathbb{R}$ such that

$$\|v - \sum a_j^\varepsilon u_j^\varepsilon\|_{\mathcal{H}} \leq 2 \frac{\delta}{\delta_1},$$

where the sum is taken over the eigenvalues of the operator \mathcal{K}^ε on the segment $[\nu - \delta_1, \nu + \delta_1]$, and $\{u_j^\varepsilon\}$ are the corresponding eigenfunctions. The coefficients a_j^ε are normalized so that $\sum |a_j^\varepsilon|^2 = 1$.

Let μ_j be an eigenvalue of the effective problem (3.18) of multiplicity \varkappa_j that is $\mu_j = \mu_{j+1} = \dots = \mu_{j+\varkappa_j-1}$, and $\{v_p(z)\}$, $p = j, \dots, j + \varkappa_j - 1$, be the eigenfunctions corresponding to μ_j . Denote

$$V_p^\varepsilon(z) = v_p(z) \chi_\varepsilon(z) + \varepsilon^{3/4} \chi_\varepsilon(z) N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v_p(z), \quad (3.25)$$

where v_p is the p th eigenfunction of the limit spectral problem (3.18), N is a solution of (3.17); $\chi_\varepsilon(z)$ is a cut-off which is equal to 1 if $|z| < \frac{\varepsilon^{-1/4}}{3} \text{dist}(0, \partial\Omega)$, equal to 0 if $|z| > \frac{\varepsilon^{-1/4}}{2} \text{dist}(0, \partial\Omega)$, and is such that

$$0 \leq \chi_\varepsilon(x) \leq 1, \quad |\nabla \chi_\varepsilon| \leq C \varepsilon^{1/4}. \quad (3.26)$$

We apply Lemma 3.6 to the operator $G^\varepsilon : H_{0,W}^1(\widetilde{\Omega}_\varepsilon) \rightarrow H_{0,W}^1(\widetilde{\Omega}_\varepsilon)$ constructed in Proposition 3.1 (see (3.13)). The normalized functions $\mathcal{V}_p^\varepsilon \equiv V_p^\varepsilon / \|V_p^\varepsilon\|_{\varepsilon,W}$ and the numbers μ_j will play the roles of $v \in \mathcal{H}$ and $\nu \in \mathbb{R}$ in Lemma 3.6. Notice that v_j need not be equal to zero on the boundary $\varepsilon^{-1/4} \partial\Omega$; the cut-off function has been introduced in order to make approximate solution (3.25) belong to the space $H_{0,W}^1(\widetilde{\Omega}_\varepsilon)$ (see (3.12)).

Lemma 3.7. *"Almost" eigenfunctions V_p^ε are almost orthonormal. Namely, the following inequalities hold:*

$$\begin{aligned} |W^\varepsilon(V_p^\varepsilon, V_q^\varepsilon) - \delta_{pq}| &\leq C \varepsilon^{1/4}, \\ |(V_p^\varepsilon, V_q^\varepsilon)_{\varepsilon,Q} - \delta_{pq}| &\leq C \varepsilon^{1/4}. \end{aligned} \quad (3.27)$$

where $W^\varepsilon(u, v)$ and $(\cdot, \cdot)_Q$ are defined by (3.10) and (3.20), respectively.

Proof. We calculate first the gradient of the function V_p^ε .

$$\nabla V_p^\varepsilon = J_{1p}^\varepsilon(z) \chi_\varepsilon(z) + \varepsilon^{3/4} J_{2p}^\varepsilon(z) + J_{3p}^\varepsilon(z) \nabla \chi_\varepsilon(z),$$

where

$$\begin{aligned} J_{1p}^\varepsilon(z) &= \nabla v_p(z) + \nabla_\zeta(N(\zeta) \cdot v_p(z)) \Big|_{\zeta=z/\varepsilon^{3/4}}; \\ J_{2p}^\varepsilon(z) &= \chi_\varepsilon(z) \nabla_z(N(\zeta) \cdot \nabla v_p(z)); \\ J_{3p}^\varepsilon(z) &= v_p(z) + \varepsilon^{3/4} N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v_p(z). \end{aligned}$$

One can show that

$$\begin{aligned} &\left| W^\varepsilon(V_p^\varepsilon, V_q^\varepsilon) - \int_{\widetilde{\Omega}_\varepsilon} a^\varepsilon(z) (\chi_\varepsilon(z))^2 J_{1p}^\varepsilon \cdot J_{1q}^\varepsilon dz \right. \\ &\quad \left. + \frac{\varkappa(0)}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega}_\varepsilon} v_p(z) v_q(z) (\chi_\varepsilon(z))^2 dz \right. \\ &\quad \left. - \varepsilon^{1/4} \int_{\Sigma_\varepsilon} q(\varepsilon^{1/4}) v_p(z) v_q(z) (\chi_\varepsilon(z))^2 d\sigma_z \right| \leq C \varepsilon^{1/4}. \end{aligned}$$

On the other hand, using Lemma 4.5, exponential decay of the eigenfunctions $v_p(z)$ and the normalization condition (3.20), we can prove that

$$\begin{aligned} & \left| \int_{\widetilde{\Omega}_\varepsilon} a^\varepsilon(z) (\chi_\varepsilon(z))^2 J_{1p}^\varepsilon \cdot J_{1q}^\varepsilon dz \right. \\ & \quad - \frac{\varkappa(0)}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega}_\varepsilon} v_p(z) v_q(z) (\chi_\varepsilon(z))^2 dz \\ & \quad \left. + \varepsilon^{1/4} \int_{\widetilde{\Sigma}_\varepsilon} q(\varepsilon^{1/4}) v_p(z) v_q(z) (\chi_\varepsilon(z))^2 d\sigma_z - \delta_{pq} \right| \leq C \varepsilon^{1/4}. \end{aligned}$$

Combining the last two estimates, we get

$$|W^\varepsilon(V_p^\varepsilon, V_q^\varepsilon) - \delta_{pq}| \leq C \varepsilon^{1/4},$$

which is the first estimate in (3.27).

The second estimate in (3.27) follows from the first one and Remark 3.4. \square

Lemma 3.8. *Let $\mathcal{V}_p^\varepsilon \equiv V_p^\varepsilon / \|V_p^\varepsilon\|_{\varepsilon, W}$ with V_p^ε defined by (3.25). Then the following estimate holds:*

$$\|G^\varepsilon \mathcal{V}_p^\varepsilon - (\mu_j)^{-1} \mathcal{V}_p^\varepsilon\|_{\varepsilon, W} \leq C_p \varepsilon^{1/4}, \quad p = i, \dots, i + \kappa_j - 1. \quad (3.28)$$

Proof. Simple transformations result in the following relations:

$$\|G^\varepsilon \mathcal{V}_p^\varepsilon - (\mu_j)^{-1} \mathcal{V}_p^\varepsilon\|_{\varepsilon, W} = \|V_p^\varepsilon\|_{\varepsilon, W}^{-1} \sup_{\|w\|_{\varepsilon, W}=1} W^\varepsilon(G^\varepsilon V_p^\varepsilon - (\mu_j)^{-1} V_p^\varepsilon, w)$$

By (3.13),

$$\begin{aligned} \|G^\varepsilon \mathcal{V}_p^\varepsilon - (\mu_j)^{-1} \mathcal{V}_p^\varepsilon\|_{\varepsilon, W} &= \frac{1}{\mu_p} \|V_p^\varepsilon\|_{\varepsilon, W}^{-1} \sup_{\|w\|_{\varepsilon, W}=1} \{ \mu_p (V_p^\varepsilon, w)_{L^2(\widetilde{\Omega}_\varepsilon)} \\ & \quad - \int_{\widetilde{\Omega}_\varepsilon} a^\varepsilon \nabla V_p^\varepsilon \cdot \nabla w dz + \frac{\varkappa(0)}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega}_\varepsilon} V_p^\varepsilon w dz - \varepsilon^{1/4} \int_{\widetilde{\Sigma}_\varepsilon} q(\varepsilon^{1/4} z) V_p^\varepsilon w d\sigma_z \} \\ &= \frac{1}{\mu_p} \|V_p^\varepsilon\|_{\varepsilon, W}^{-1} \sup_{\|w\|_{\varepsilon, W}=1} \{ I_1^\varepsilon + I_2^\varepsilon + \varepsilon^{3/4} I_3^\varepsilon \}. \end{aligned}$$

Here

$$\begin{aligned} I_1^\varepsilon &= \mu_p \int_{\widetilde{\Omega}_\varepsilon} \chi_\varepsilon(z) v_p(z) w(z) dz - \frac{1}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega}_\varepsilon} (\varkappa(\varepsilon^{1/4} z) - \varkappa(0)) \chi_\varepsilon(z) v_p(z) w(z) dz \\ & \quad - \int_{\widetilde{\Omega}_\varepsilon} a(\zeta) \left(\nabla v_p(z) + \nabla_\zeta(N(\zeta) \cdot \nabla v_p(z)) \right) \cdot \nabla w \chi_\varepsilon(z) \Big|_{\zeta=z/\varepsilon^{3/4}} dz; \end{aligned}$$

$$\begin{aligned}
I_2^\varepsilon &= \frac{1}{\sqrt{\varepsilon}} \int_{\widetilde{\Omega_\varepsilon}} \varkappa(\varepsilon^{1/4} z) v_p(z) \chi_\varepsilon(z) w(z) dz - \varepsilon^{1/4} \int_{\widetilde{\Sigma_\varepsilon}} q(\varepsilon^{1/4} z) v_p(z) \chi_\varepsilon(z) w(z) d\sigma; \\
I_3^\varepsilon &= \mu_p \int_{\widetilde{\Omega_\varepsilon}} \chi_\varepsilon(z) N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v_p(z) w(z) dz \\
&\quad - \int_{\widetilde{\Omega_\varepsilon}} a(\zeta) \nabla \chi_\varepsilon(z) \cdot \nabla w v_p(z) \Big|_{\zeta=z/\varepsilon^{3/4}} dz \\
&\quad - \int_{\widetilde{\Omega_\varepsilon}} a(\zeta) \nabla_z (\chi_\varepsilon(z) N(\zeta) \cdot \nabla v_p(z)) \cdot \nabla w \Big|_{\zeta=z/\varepsilon^{3/4}} dz \\
&\quad + \varepsilon^{1/4} \varkappa(0) \int_{\widetilde{\Omega_\varepsilon}} \chi_\varepsilon(z) N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v_p(z) w(z) dz \\
&\quad - \varepsilon^{1/4} \int_{\widetilde{\Sigma_\varepsilon}} q(\varepsilon^{1/4} z) \chi_\varepsilon(z) N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v_p(z) w d\sigma.
\end{aligned}$$

Integrating by parts in the last integral in I_1^ε , taking into account **(H4)**, (3.17) and (3.26), we obtain

$$\begin{aligned}
I_1^\varepsilon &= \mu_p \int_{\widetilde{\Omega_\varepsilon}} \chi_\varepsilon(z) v_p(z) w(z) dz - \int_{\widetilde{\Omega_\varepsilon}} (z^T Q z) \chi_\varepsilon(z) v_p(z) w(z) dz \\
&\quad + \int_{\widetilde{\Omega_\varepsilon}} \operatorname{div}_z (a(\zeta) (I + \nabla_\zeta N(\zeta)) \nabla v_p(z)) \Big|_{\zeta=z/\varepsilon^{3/4}} w(z) \chi_\varepsilon(z) dz + O(\varepsilon^{1/4}), \quad \varepsilon \rightarrow 0.
\end{aligned}$$

Here we have also used Lemma 3.3 and the fact that $\|w\|_{\varepsilon, W} = 1$.

Bearing in mind the definition of the effective diffusion (3.19) and (3.18), by virtue of Lemma 4.5, one has

$$|I_1^\varepsilon| \leq C \varepsilon^{1/4} \|w\|_{H^1(\mathbb{R}^d)}. \quad (3.29)$$

By Lemma 4.2,

$$|I_2^\varepsilon| \leq C_2 \varepsilon^{1/4} \|v_p\|_{H^1(\mathbb{R}^d)} \|w\|_{H^1(\mathbb{R}^d)}. \quad (3.30)$$

Using the boundedness of a_{ij} and the regularity properties of N, v_p, χ_ε , one can show that

$$|I_3^\varepsilon| \leq C_3 \|\nabla v_p\|_{L^2(\mathbb{R}^d)} \|w\|_{H^1(\mathbb{R}^d)}. \quad (3.31)$$

Using Lemma 3.7 we see that for small enough ε ,

$$\|V_p^\varepsilon\|_{\varepsilon, W}^2 \geq \frac{1}{2}. \quad (3.32)$$

Finally, combining (3.29)-(3.32) we obtain the desired estimate (3.28). Lemma 3.8 is proved. \square

By Lemma 3.6, in view of the estimate obtained in Lemma 3.8, for any eigenvalue μ_j of the effective problem (3.18) there exists an eigenvalue of the original problem such that

$$|\mu_{q(j)}^\varepsilon - \mu_j| \leq C_j \varepsilon^{1/4}, \quad (3.33)$$

where $q(j)$ might depend on ε .

Moreover, letting δ_1 in the statement of Lemma 3.6 be equal to $\Theta_j \varepsilon^{1/4}$ (the constant Θ_j will be chosen below), we conclude that there exists a $K_j(\varepsilon) \times \kappa_j$ constant matrix α^ε such that

$$\left\| \mathcal{V}_p^\varepsilon - \sum_{k=J_j}^{J_j+K_j(\varepsilon)-1} \alpha_{kp}^\varepsilon v_k^\varepsilon \right\|_{\varepsilon, W} \leq 2 \frac{C \varepsilon^{1/4}}{\delta_1} \leq C_j (\Theta_j)^{-1} \quad p = j, \dots, j + \kappa_j - 1, \quad (3.34)$$

here μ_k^ε , $k = J_j(\varepsilon), \dots, J_j(\varepsilon) + K_j(\varepsilon) - 1$, are all the eigenvalues of operator $(G^\varepsilon)^{-1}$ which satisfy the estimate

$$|\mu_k^\varepsilon - \mu_j| \leq \Theta_j \varepsilon^{1/4}. \quad (3.35)$$

Since the eigenvalues μ_j do not depend on ε , one can choose constants $\varepsilon_j > 0$ so that the intervals $(\mu_j - \Theta_j \varepsilon^{1/4}, \mu_j + \Theta_j \varepsilon^{1/4})$ and $(\mu_i - \Theta_i \varepsilon^{1/4}, \mu_i + \Theta_i \varepsilon^{1/4})$ do not intersect if $\mu_j \neq \mu_i$ and $\varepsilon < \min(\varepsilon_j, \varepsilon_i)$. Then the sets of eigenvalues $\{\mu_k^\varepsilon\}$ related to different μ_j in (3.35) do not intersect for sufficiently small ε .

In the following statement we prove that $K_J(\varepsilon) \geq \kappa_j$.

Lemma 3.9. *The columns of the matrix α^ε , that is the vectors $\{\alpha_{\cdot p}^\varepsilon\}_{p=J(j)}^{J(j)+\kappa_j-1}$ of length $K_J(\varepsilon)$ are linearly independent. As a consequence, $K_J(\varepsilon) \geq \kappa_j$.*

Proof. A simple transformation gives

$$\begin{aligned} W^\varepsilon(\mathcal{V}_p^\varepsilon, \mathcal{V}_q^\varepsilon) &= W^\varepsilon\left(\mathcal{V}_p^\varepsilon - \sum_{k=J_j}^{J_j+K_j(\varepsilon)-1} \alpha_{kp}^\varepsilon v_k^\varepsilon, \mathcal{V}_p^\varepsilon\right) + \\ &+ W^\varepsilon\left(\sum_{k=J_j}^{J_j+K_j(\varepsilon)-1} \alpha_{kp}^\varepsilon v_k^\varepsilon, \mathcal{V}_q^\varepsilon - \sum_{k=J_j}^{J_j+K_j(\varepsilon)-1} \alpha_{kq}^\varepsilon v_k^\varepsilon\right) + \sum_{k=J_j}^{J_j+K_j(\varepsilon)-1} \alpha_{kp}^\varepsilon \alpha_{kq}^\varepsilon. \end{aligned}$$

Taking estimates (3.27) and (3.34) into account, we obtain

$$\left| \sum_{k=J_j}^{J_j+K_j(\varepsilon)-1} \alpha_{kp}^\varepsilon \alpha_{kq}^\varepsilon - \delta_{p,q} \right| \leq C \Theta_j^{-1}, \quad p, q = J(j), \dots, J(j) + \kappa_j - 1,$$

and, in other words,

$$|(\alpha_{\cdot p}^\varepsilon)^T \alpha_{\cdot q}^\varepsilon - \delta_{p,q}| \leq C \Theta_j^{-1}, \quad p, q = J(j), \dots, J(j) + \kappa_j - 1, \quad (3.36)$$

where $\alpha_{\cdot p}^\varepsilon$ denotes a p th column in the matrix α^ε . The last inequality means that the vectors $\{\alpha_{\cdot p}^\varepsilon\}_{p=J(j)}^{J(j)+\kappa_j-1}$ are asymptotically orthonormal, as Θ_j grows to infinity. This property implies the linear independence of the vectors $\{\alpha_{\cdot p}^\varepsilon\}$

for sufficiently large Θ_j . Indeed, assume that $\{\alpha_p^\varepsilon\}_{p=J(j)}^{J(j)+\kappa_j-1}$ are not linearly independent. Then there exist constants $c_{J(j)}, \dots, c_{J(j)+\kappa_j-1}$ such that

$$\sum_{k=J(j)}^{J(j)+\kappa_j-1} c_k \alpha_k^\varepsilon = 0.$$

Without loss of generality we assume that $c_{J(j)} = 1 \geq \max_k |c_k|$. Then

$$\alpha_{\cdot, J(j)}^\varepsilon + \sum_{k>J(j)} c_k \alpha_k^\varepsilon = 0.$$

Multiplying the last equality by $\alpha_{\cdot, J(j)}^\varepsilon$ and using (3.36) we obtain the inequality

$$|(\alpha_{\cdot, J(j)}^\varepsilon)^T \alpha_{\cdot, J(j)}^\varepsilon| \leq C_j \Theta_j^{-1},$$

that contradicts (3.36) if Θ_j^{-1} is sufficiently small. Thus, the vectors $\{\alpha_p^\varepsilon\}_{p=J(j)}^{J(j)+\kappa_j-1}$ of length $K_J(\varepsilon)$ are linearly independent. Obviously, it is possible only in the case $K_J(\varepsilon) \geq \kappa_j$. \square

Lemma 3.10. *For any q , $0 < m \leq \mu_q^\varepsilon \leq M_q$.*

Proof. The estimate from below is the immediate consequence of the boundedness of the operator G^ε constructed in Proposition 3.1.

To obtain an upper bound for μ_q^ε , we recall estimate (3.33). For any j , there exists an eigenvalue of problem (3.7) converging to the j th eigenvalue of the effective problem. Namely, the estimate holds

$$|\mu_{q_\varepsilon(j)}^\varepsilon - \mu_{J(j)}| \leq C_j \varepsilon^{1/4},$$

where $J(j) = \min\{i \in \mathbb{Z}^+ : \mu_i = \mu_j\}$. Obviously, $q_\varepsilon(j) \geq J(j)$. Thus,

$$\mu_{J(j)}^\varepsilon \leq \mu_{q_\varepsilon(j)}^\varepsilon \leq \mu_{J(j)} + C_j \varepsilon^{1/4}$$

that implies the desired bound. \square

Our next goal is to prove that any accumulation point of the sequence μ_q^ε , as $\varepsilon \rightarrow 0$, is an eigenvalue of (3.18).

Lemma 3.11. *If, up to a subsequence, $\mu_q^\varepsilon \rightarrow \mu^*$, as $\varepsilon \rightarrow 0$, then μ^* is an eigenvalue of the effective spectral problem (3.18).*

Proof. Since μ_q^ε is bounded, then

$$\|v_q^\varepsilon\|_{\varepsilon, W} \leq C_q$$

with $\|\cdot\|_{\varepsilon, W}$ defined in (3.14). In view of Lemmata 3.3 and 4.4, the eigenfunction v_q^ε (extended to the whole \mathbb{R}^d) converges weakly in $H^1(\mathbb{R}^d)$ and strongly in $L^2(\mathbb{R}^d)$ to some function v^* . To prove that (μ^*, v^*) is an eigenpair of the effective problem

(3.18), we pass to the limit in the integral identity (3.9). Using standard two-scale convergence arguments we obtain

$$\int_{\mathbb{R}^d} a^{\text{eff}} \nabla v^* \cdot \nabla w \, dz + \int_{\mathbb{R}^d} (z^T Q z) v^* w \, dz = \mu^* \int_{\mathbb{R}^d} v^* w \, dz, \quad w \in H^1(\mathbb{R}^d).$$

The last equality is the weak formulation of (3.18). Since $\mu_q^\varepsilon \rightarrow \mu^*$, as $\varepsilon \rightarrow 0$, then considering (3.9) and (3.11) we conclude that $\lim_{\varepsilon \rightarrow 0} \|v_q^\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)}^2 = \mu^*$. Using the strong convergence of v_q^ε in $L^2(\mathbb{R}^d)$, we see that $\|v^*\|_{L^2(\mathbb{R}^d)}^2 \geq \mu^*$. By Lemma 3.4 we have $\mu^* > 0$. Therefore, $v^* \neq 0$. This completes the proof. \square

Lemma 3.12. *Let μ_j be the j th eigenvalue of problem (3.18) of multiplicity κ_j , that is $\mu_j = \mu_{j+1} = \dots = \mu_{j+\kappa_j-1}$. Then there exist exactly κ_j eigenvalues of the original problem (2.1) converging to it.*

Proof. First, we prove that there are not more than κ_j eigenvalues of problem (3.7) converging to μ_j . Assume that there exist $\kappa_j + 1$ eigenvalues $\mu_{J_k^\varepsilon(j)}$ such that

$$\mu_{J_k^\varepsilon(j)} \rightarrow \mu_j, \quad k = 1, \dots, \kappa_j + 1.$$

By Lemma 3.11, the corresponding eigenfunctions $v_{J_k^\varepsilon(j)}$, extended to the whole \mathbb{R}^d , converge weakly in $H^1(\mathbb{R}^d)$ and strongly in $L^2(\mathbb{R}^d)$ to the eigenfunctions v_k^* of the effective problem (3.18), $k = 1, \dots, \kappa_j + 1$. Passing to the limit in the normalization condition (3.11) yields

$$(v_i^*, v_k^*)_{L^2(\mathbb{R}^d)} = \frac{1}{|Y|_d} \mu_i \delta_{ik}, \quad i, k = 1, \dots, \kappa_j + 1.$$

Therefore, eigenfunctions $\{v_k^*\}_{k=1}^{\kappa_j+1}$ corresponding to μ_j are linearly independent. Recalling that the multiplicity of μ_j is κ_j , we arrive at contradiction. Thus, there are not more than κ_j eigenvalues of problem (3.7) converging to μ_j .

On the other hand, by Lemma 3.9, there exist at least κ_j eigenvalues of (3.7) converging to μ_j of multiplicity κ_j . Lemma 3.12 is proved. \square

Combining Lemmata 3.10–3.12 completes the proof of the first statement of Theorem 3.1.

We turn to the proof of the second statement in Theorem 3.1.

First of all, let us notice that the orthogonality and normalization condition (3.24) follows directly from Lemma 3.7 and the exponential decay of $v_k(z)$ as eigenfunctions of the harmonic oscillator.

In order to prove estimate (3.22), we recall the estimate obtained in Lemma 3.8 and apply the estimate in Lemma 3.6 with $\delta_1 = c_j$, c_j being a sufficiently small constant. This estimate reads

$$\left\| \mathcal{V}_p^\varepsilon - \sum_{\mu_k^\varepsilon \in S(j, \varepsilon)} \alpha_{kp}^\varepsilon v_k^\varepsilon \right\|_{\varepsilon, W} \leq 2 \frac{C \varepsilon^{1/4}}{\delta_1} \leq C_j \varepsilon^{1/4} \quad p = j; \dots, j + \kappa_j - 1,$$

where $S(j, \varepsilon)$ is the set of eigenvalues μ_k^ε satisfying the estimate

$$|\mu_k^\varepsilon - \mu_j| \leq c_j;$$

the constant matrix α^ε is such that

$$|(\alpha_p^\varepsilon)^T \alpha_{q, \varepsilon}^\varepsilon - \delta_{p,q}| \leq C_j \varepsilon^{1/4}, \quad p, q = J(j), \dots, J(j) + \kappa_j - 1, \quad (3.37)$$

From the first statement of Theorem 3.1 we deduce that the set $S(j, \varepsilon)$ coincides with the set of eigenvalues $\{\mu_k^\varepsilon\}_{k=J(j)}^{J(j)+\kappa_j-1}$, for sufficiently small ε . Therefore,

$$\left\| \mathcal{V}_p^\varepsilon - \sum_{k=J(j)}^{J(j)+\kappa_j-1} \alpha_{kp}^\varepsilon v_k^\varepsilon \right\|_{\varepsilon, W} \leq C_j \varepsilon^{1/4} \quad p = j; \dots, j + \kappa_j - 1, \quad (3.38)$$

with a constant $\kappa_j \times \kappa_j$ matrix α^ε which satisfies inequality (3.37).

It remains to use the following simple statement.

Lemma 3.13. *For any $n \times n$ matrix A satisfying an equality*

$$\|A^T A - \mathbb{I}\|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)} = \gamma \in (0, 1),$$

there exists a unitary matrix B such that

$$\|AB - \mathbb{I}\|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)} \leq \gamma;$$

here \mathbb{I} is a unit matrix, and

$$\|D\|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)} = \sup_{\substack{\xi \in \mathbb{R}^n \\ \|\xi\|=1}} \|D\xi\|.$$

We omit the proof of this lemma which can be found in [5]. According to (3.37) and Lemma 3.13, there exists a unitary $\kappa_j \times \kappa_j$ matrix β^ε such that

$$\|\alpha^\varepsilon \beta^\varepsilon - \mathbb{I}\|_{\mathcal{L}(\mathbb{R}^{\kappa_j}, \mathbb{R}^{\kappa_j})} \leq C_j \varepsilon^{1/4}. \quad (3.39)$$

Taking into account Lemma 3.7, estimates (3.38), (3.39), one can show that

$$\left\| v_p^\varepsilon - \sum_{k=J(j)}^{J(j)+\kappa_j-1} \beta_{kp}^\varepsilon V_k^\varepsilon \right\|_{\varepsilon, W} \leq C_j \varepsilon^{1/4}, \quad p = J(j), \dots, J(j) + \kappa_j - 1.$$

Due to the exponential decay of the eigenfunctions $v_k(z)$ defined in (3.18), one can replace V_k^ε defined by (3.25) with (3.23). Then, by Lemma 3.3, similar estimate holds for $\|\cdot\|_{\varepsilon, Q}$ norm. Theorem 3.1 is proved. \square

Bearing in mind the result obtained in Theorem 3.1, we formulate the main result of the present paper characterizing the asymptotic behaviour of eigenpairs $(\lambda_j^\varepsilon, u_j^\varepsilon)$ of problem (2.1).

Theorem 3.2. *Let conditions (H1)-(H4) be fulfilled. If $(\lambda_j^\varepsilon, u_j^\varepsilon)$ stands for the j th eigenpair of problem (2.1), then for any j , the following representation takes place:*

$$\lambda_j^\varepsilon = \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} q(0) + \frac{\mu_j^\varepsilon}{\sqrt{\varepsilon}}, \quad u_j^\varepsilon(x) = v_j^\varepsilon\left(\frac{x}{\varepsilon^{1/4}}\right),$$

where the eigenpairs $(\mu_j^\varepsilon, v_j^\varepsilon(z))$ of problem (3.7) are such that

1. For each $j = 1, 2, \dots$, there exist $\varepsilon_j > 0$ and a constant c_j such that

$$|\mu_j^\varepsilon - \mu_j| \leq c_j \varepsilon^{1/4}, \quad \varepsilon \in (0, \varepsilon_j),$$

where μ_j is an eigenvalue of the harmonic oscillator operator (3.18).

2. Let μ_j be an eigenvalue of (3.18) of multiplicity κ_j , that is $\mu_j = \dots = \mu_{j+\kappa_j-1}$. Then, there exists a unitary $\kappa_j \times \kappa_j$ matrix β^ε such that

$$\left\| v_p^\varepsilon - \sum_{k=J(j)}^{J(j)+\kappa_j-1} \beta_{kp}^\varepsilon \tilde{V}_k^\varepsilon \right\|_{\varepsilon, Q} \leq C_j \varepsilon^{1/4}, \quad p = J(j), \dots, J(j) + \kappa_j - 1,$$

where

$$\tilde{V}_k^\varepsilon = v_k(z) + \varepsilon^{3/4} N\left(\frac{z}{\varepsilon^{3/4}}\right) \cdot \nabla v_k(z).$$

Here the vector function $N(\zeta)$ solve problem (3.17); eigenfunctions $v_k(z)$ of the limit problem are defined in (3.18); the norm $\|\cdot\|_{\varepsilon, Q}$ is defined in (3.14).

4 Auxiliary results

Lemma 4.1. *For any $w^\varepsilon(x) \in H_0^1(\Omega_\varepsilon, \partial\Omega)$ the following estimate holds*

$$\left| \frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \int_{\Omega_\varepsilon} |w^\varepsilon|^2 dx - \int_{\Sigma_\varepsilon} |w^\varepsilon|^2 d\sigma \right| \leq C \|w^\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla w^\varepsilon\|_{L^2(\Omega_\varepsilon)}$$

with a constant C independent of ε .

Proof. Introduce a Y -periodic vector function $\chi(y)$ as a solution of the following problem on the periodicity cell Y :

$$\begin{cases} -\operatorname{div}_y \chi = \frac{|\Sigma^0|_{d-1}}{|Y|_d}, & y \in Y, \\ (\chi, n) = -1, & y \in \Sigma^0. \end{cases}$$

Notice that χ is a smooth function. Then

$$-\varepsilon \operatorname{div}_x \chi\left(\frac{x}{\varepsilon}\right) = \frac{|\Sigma^0|_{d-1}}{|Y|_d}.$$

Multiplying the last equality by $|w^\varepsilon|^2$ and integrating by parts over Ω_ε yields

$$\frac{1}{\varepsilon} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \int_{\Omega_\varepsilon} |w^\varepsilon|^2 dx - \int_{\Sigma_\varepsilon} |w^\varepsilon|^2 d\sigma = \int_{\Omega_\varepsilon} \left(\chi\left(\frac{x}{\varepsilon}\right), \nabla |w^\varepsilon|^2 \right) dx$$

that easily implies the desired estimate. Lemma is proved. \square

Lemma 4.2. Let $\widetilde{\Omega}_\varepsilon = \varepsilon^{-\alpha} \Omega_\varepsilon$, $\widetilde{\Sigma}_\varepsilon = \varepsilon^{-\alpha} \Sigma_\varepsilon$. Then, for $\psi(z) \in H_0^1(\widetilde{\Omega}_\varepsilon, \varepsilon^{-\alpha} \partial \Omega)$ and $\varphi \in C^1(\mathbb{R}^d)$, the following estimate holds

$$\left| \frac{1}{\varepsilon^{1-\alpha}} \frac{|\Sigma^0|_{d-1}}{|Y|_d} \int_{\widetilde{\Omega}_\varepsilon} \varphi(\varepsilon^\alpha z) |\psi(z)|^2 dx - \int_{\widetilde{\Sigma}_\varepsilon} \varphi(\varepsilon^\alpha z) |\psi(z)|^2 d\sigma \right| \leq C \|\psi\|_{L^2(\widetilde{\Omega}_\varepsilon)} \|\nabla \psi\|_{L^2(\widetilde{\Omega}_\varepsilon)}.$$

with some constant C independent of ε .

Lemma 4.2 is proved in the same way as Lemma 4.1.

Lemma 4.3. Suppose two nonnegative functions $f_1, f_2 \in C^3(\bar{B})$, defined on a bounded domain B , are such that $x = 0$ is the global minimum point for both of them, and $f_1(0) = f_2(0) = 0$. Moreover, assume that

$$H(f_k)(0) \geq \alpha I, \quad \alpha > 0,$$

where $H(f_k)$ is the Hessian matrix of f_k , $k = 1, 2$.

Then there exists a constant C such that

$$C f_1 \leq f_2 \leq C^{-1} f_1.$$

Proof. Assume that there exists a sequence $x_j \in B$ such that

$$\frac{f_1(x_j)}{f_2(x_j)} \rightarrow 0, \quad j \rightarrow \infty.$$

Since f_2 is bounded, then $f_1(x_j) \rightarrow 0$, as $j \rightarrow \infty$. And, consequently, $x_j \rightarrow 0$, as $j \rightarrow \infty$. Recalling that $H(f_1)(0)$ is bounded from below, we arrive at contradiction. Lemma is proved. \square

Lemma 4.4. Compactness result

Denote

$$H_Q^1(\mathbb{R}^d) = \left\{ w \in H^1(\mathbb{R}^d) : \|w\|_Q^2 = \int_{\mathbb{R}^d} |\nabla w|^2 dz + \int_{\mathbb{R}^d} (z^T Q z) |w|^2 dz < \infty \right\}.$$

Then $H_Q^1(\mathbb{R}^d)$ is compactly imbedded into $L^2(\mathbb{R}^d)$. In other words, any $\{v_n\} \subset H^1(\mathbb{R}^d)$ such that $\|v_n\|_Q \leq C$, converges strongly along a subsequence in $L^2(\mathbb{R}^d)$.

Proof. Obviously, v_n , up to a subsequence, converges weakly in $L^2(\mathbb{R}^d)$ to some function v^* , $n \rightarrow \infty$. Let us prove that $\|v_n\|_{L^2(\mathbb{R}^d)} \rightarrow \|v^*\|_{L^2(\mathbb{R}^d)}$, as $n \rightarrow \infty$.

Since

$$\int_{\mathbb{R}^d} (z^T Q z) |v_n|^2 dz \leq C,$$

one can show that for any $\delta > 0$, there exists a ball $B_{R(\delta)}(0)$ such that

$$\int_{\mathbb{R}^d \setminus B_{R(\delta)}(0)} |v_n|^2 dz \leq \delta.$$

Without loss of generality we assume that $\|v_n\|_{L^2(\mathbb{R}^d)} = 1$. Then

$$\|v_n\|_{L^2(B_{R(\delta)}(0))}^2 = 1 - \|v_n\|_{L^2(\mathbb{R}^d) \setminus B_{R(\delta)}(0)}^2 \geq 1 - \delta^2. \quad (4.1)$$

Since $\|v_n\|_{H^1(B_{R(\delta)}(0))} \leq C$, then $\|v_n - v^*\|_{L^2(B_{R(\delta)}(0))} \rightarrow 0$, as $n \rightarrow \infty$. Passing to the limit in (4.1), we have

$$\|v^*\|_{L^2(\mathbb{R}^d)} \geq \|v^*\|_{L^2(B_{R(\delta)}(0))}^2 \geq 1 - \delta^2.$$

On the other hand,

$$\|v^*\|_{L^2(\mathbb{R}^d)} \leq \liminf_{n \rightarrow \infty} \|v_n\|_{L^2(\mathbb{R}^d)} = 1.$$

Combining the last two inequalities yields $\|v^*\|_{L^2(\mathbb{R}^d)} = 1$. Lemma is proved. \square

Lemma 4.5. Mean-value theorem

Let $\Phi \in L^2(Y)$ be such that $\int_Y \Phi dy = 0$, and $V \in C^1(\mathbb{R}^d)$ satisfy the estimate

$$|D^k V(z)| \leq C e^{-\gamma_0 |z|^2}, \quad \gamma > 0, \quad k = 0, 1. \quad (4.2)$$

Denote by $\chi(x)$ a cut-off which is equal to 1 if $|x| < \frac{1}{3} \text{dist}(0, \partial\Omega)$, equal to 0 if $|x| > \frac{1}{2} \text{dist}(0, \partial\Omega)$, and is such that

$$0 \leq \chi(x) \leq 1, \quad |\nabla \chi| \leq C. \quad (4.3)$$

Then the following estimate holds:

$$\left| \int_{\Omega_\varepsilon} \Phi\left(\frac{x}{\varepsilon}\right) V\left(\frac{x}{\varepsilon^\alpha}\right) \chi(x) W\left(\frac{x}{\varepsilon^\alpha}\right) dx \right| \leq C \varepsilon^{1-\alpha} \varepsilon^{d\alpha} \|\Phi\|_{L^2(Y)} \|W\|_{H^1(\mathbb{R}^d)}$$

for any $W \in H^1(\mathbb{R}^d)$.

Proof. Since $\int_Y \Phi dy = 0$, then there exists a periodic vector function $\varphi(y)$ such that

$$\begin{cases} -\text{div}_y \varphi(y) = \Phi(y), & y \in Y, \\ (\varphi, n) = 0, & y \in \Sigma^0, \end{cases}$$

and $\|\varphi\|_{L^2(Y)} \leq C \|\Phi\|_{L^2(Y)}$. Changing the variables we have

$$-\varepsilon \text{div} \varphi\left(\frac{x}{\varepsilon}\right) = \Phi\left(\frac{x}{\varepsilon}\right).$$

Multiplying the last equation by $V\left(\frac{x}{\varepsilon^\alpha}\right) \chi_\varepsilon(x) W\left(\frac{x}{\varepsilon^\alpha}\right)$, integrating by parts over Ω_ε and using (4.2), (4.3) we get

$$\begin{aligned} & \left| \int_{\Omega_\varepsilon} \Phi\left(\frac{x}{\varepsilon}\right) V\left(\frac{x}{\varepsilon^\alpha}\right) \chi(x) W\left(\frac{x}{\varepsilon^\alpha}\right) dx \right| \\ &= \varepsilon \left| \int_{\Omega_\varepsilon} \varphi\left(\frac{x}{\varepsilon}\right) \cdot \nabla \left[V\left(\frac{x}{\varepsilon^\alpha}\right) \chi(x) W\left(\frac{x}{\varepsilon^\alpha}\right) \right] dx \right| \\ &\leq C \varepsilon^{1-\alpha} \varepsilon^{d\alpha} \int_{\widetilde{\Omega_\varepsilon}} \left| \varphi\left(\frac{z}{\varepsilon^{1-\alpha}}\right) \right| e^{-\gamma_0 |z|^2} [|W| + |\nabla W|] dz \\ &\leq C \varepsilon^{1-\alpha} \varepsilon^{d\alpha} \|W\|_{H^1(\mathbb{R}^d)} \left(\int_{\widetilde{\Omega_\varepsilon}} \left| \varphi\left(\frac{z}{\varepsilon^{1-\alpha}}\right) \right|^2 e^{-2\gamma_0 |z|^2} dz \right)^{1/2} \\ &\leq C \varepsilon^{1-\alpha} \varepsilon^{d\alpha} \|W\|_{H^1(\mathbb{R}^d)} \|\varphi\|_{L^2(Y)} \left(\int_{\mathbb{R}^d} e^{-2\gamma_0 |z|^2} dz \right)^{1/2}. \end{aligned}$$

The integral in the parentheses converges. Lemma 4.5 is proved. \square

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